Week 3:
Expected Utility Theory

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Outline

Review Quiz

Interactive quiz on Vevox

Quick review of Week 2 material

Please open Vevox and enter the session code

Expected Utility Theory

Behavioural Finance vs. Expected Utility Theory

Traditional finance assumes agents are "rational":

Rational Beliefs

- Update beliefs correctly using Bayes' law
- Process all available information

Rational Choices

- Maximize expected utility
- Consistent preferences
- Independent decisions

So far: how people form beliefs

Today: how people evaluate risky prospects

Rational Preferences

Certain conditions are commonly imposed on preferences: axioms of rationality.

Suppose a person is confronted with the choice between two outcomes, A and B:

1. Either $A \succeq B$ or $B \succeq A$ (Completeness)
2. If we had C: $A \succeq B; B \succeq C \Rightarrow A \succeq C$ (Transitivity)
3. $A \succeq B \succeq C \Rightarrow \exists p \in [0,1] : B \sim p \cdot A + (1-p) \cdot C$ (Continuity)
4. $A \succeq B \Rightarrow pA + (1-p)C \succeq pB + (1-p)C$ (Independence)

Utility Function

Example: we need to choose between A = {2 bread loafs and 1 water bottle}, and B = {1 bread loaf and 2 water bottles}.

The utility function is a function that represents the preferences or binary relation $\succeq$ such that:

$$A \succeq B \text{ if and only if } U(A) \geq U(B)$$

$U(\cdot)$ assigns a real number $u(x_i)$ to each possible outcome x_i:

$$\text{2 bread} + \text{1 water} \quad \succeq \quad \text{1 bread} + \text{2 water}$$

if and only if

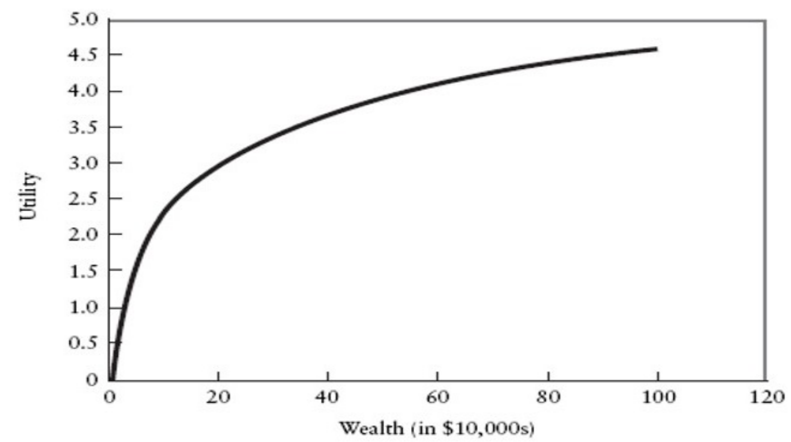
$$U(\text{2 bread} + \text{1 water}) \quad \geq \quad U(\text{1 bread} + \text{2 water})$$

- The utility function is ordinal
- Individual will choose the bundle that maximizes her utility (given the budget constraint)

Utility Maximization

If there is a single good, utility functions are often defined in relation to wealth w.

We can have different forms, but a popular one is: $u(w) = \ln(w)$ over wealth w



Expected Utility Theory

EUT was developed by von Neumann and Morgenstern to define rational behaviour under uncertainty.

It states how individuals *should* act when confronted with decision-making under risk (normative theory).

It is set up to deal with risk, rather than uncertainty:

Risk We know what the outcomes could be, and probabilities are objectively known

Uncertainty List of possible outcomes and/or probabilities are not objectively known

Prospects

A prospect or lottery is a series of wealth outcomes, each of which is associated with a probability.

Given:

- W: the current wealth of the decision maker
- $U(\cdot)$: utility function
- $X = \{x_1, x_2, \ldots, x_n\}$: a finite, fixed set of all possible outcomes from a prospect
- $p_1, p_2, \ldots, p_n$: probability corresponding to each outcome x_i, such that $p_i \in [0, 1]$ and $\sum_i p_i = 1$

$\rightarrow$ Lottery or prospect L:

$$L = (x_1, p_1; x_2, p_2; \ldots; x_n, p_n)$$

Prospects: Example

Let's see an example.

We have only 2 outcomes: low wealth $w_L = \$500$ with probability 0.8 and high wealth $w_H = \$2{,}000$ with probability 0.2.

This can be written as:

$$\begin{aligned} L &= (500, 0.8; 2{,}000, 0.2), \text{ or} \\ L &= (0.8, 500, 2{,}000) \end{aligned}$$

Expected utility theory comes from a series of assumptions (axioms) on these prospects.

Expected Utility Theory

The utility function has the "expected utility form" (von Neumann and Morgenstern) if we can assign a real number $u(x_i)$ to each possible outcome x_i, such that for prospect A we can write:

$$U(A) = \sum p_i \cdot u(W + x_i)$$

Based on assumptions such as ordering, transitivity, independence (and others), when choices over risky prospects are to be made, people should act as if they are maximizing the expected utility levels:

$\Rightarrow$ Individuals choose the gamble with the highest expected utility.

Two features:

- Linearity in probabilities
- Utility depends on the final wealth level in each state

Example of Transitivity and Completeness of Preferences

Consider four prospects P1 to P4:

- P1: $(0.8, 500, 2{,}000)$
- P2: $(0.8, 500)$
- P3: $(0.7, 300, 2{,}100)$
- P4: $(0.5, 600, 2{,}000)$

- Completeness would say (for example) that: $P1 \succeq P3$ and $P4 \succeq P1$
- Transitivity says that: $P4 \succeq P3$

Example of Logarithmic Utility Function

Let's assume $u(w) = \ln(w)$.

Which prospect do we choose between the two:

- P1: $(0.4, 50, 1{,}000)$
- P2: $(0.5, 100, 1{,}000)$

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Which prospect do we choose between the two:

- P1: $(0.4, 50, 1{,}000)$
- P2: $(0.5, 100, 1{,}000)$

- For P1: $U(P1) = 0.4 \times \ln(50) + 0.6 \times \ln(1{,}000) = 3.41$
- For P2: $U(P2) = 0.5 \times \ln(100) + 0.5 \times \ln(1{,}000) = 3.45$

- Therefore, for this utility function:

$$P2 \succeq P1 \quad \text{and} \quad U(P2) \geq U(P1)$$

Attitudes to Risk

Expected Utility Theory: Attitudes to Risk

Most people are willing to assume risk if they are compensated for it.

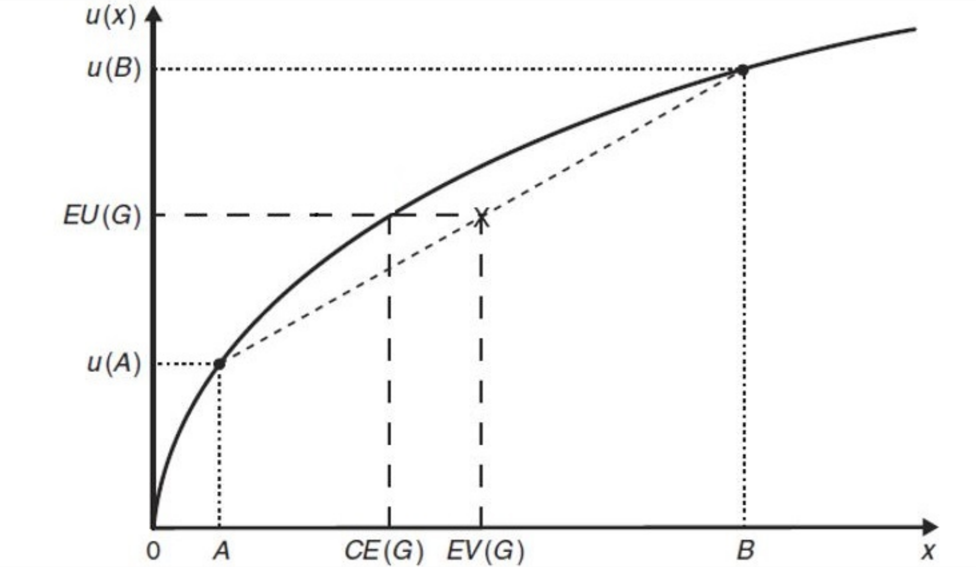
An individual that would accept a sure amount that is lower than the expected value of the lottery is said to be risk-averse:

$$CE_L < EV(L)$$

where CE_L is the certainty equivalent of the lottery L: $U(CE_L) = U(L)$, thus $U(L) < U(EV(L))$

$\rightarrow$ Concave utility function demonstrates risk aversion

Risk Aversion Illustration



Consider prospect P1: $(0.4, 5, 100)$

Using logarithmic utility function, we have the expected utility:

$$\begin{aligned} U(P1) &= 0.40 \times u(5) + 0.60 \times u(100) \\ &= 0.40 \times (1.6094) + 0.60 \times (4.6052) = 3.41 \end{aligned}$$

The utility of the expected value is instead:

$$\begin{aligned} EV(P1) &= 0.40 \times (5) + 0.60 \times (100) = 62 \\ U(EV(P1)) &= \ln(62) = 4.1271 \end{aligned}$$

$\rightarrow U(L) < U(EV(L))$

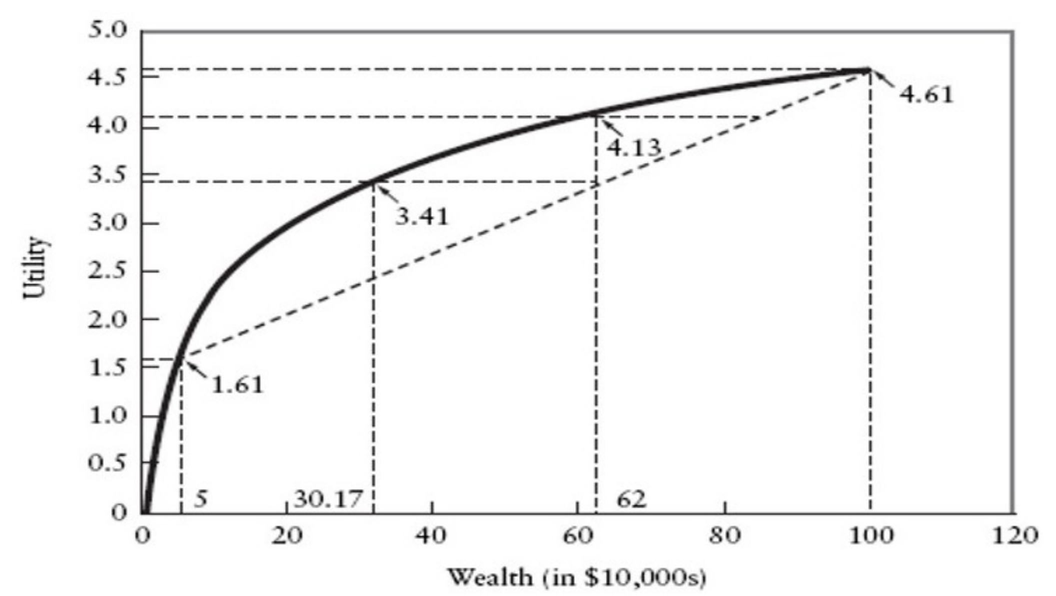
Certainty Equivalent (I)

Certainty equivalent: wealth level which leads decision-maker to be indifferent between a prospect and a certain wealth level:

$$w \text{ such that: } U(P1) = 3.41 = U(w)$$

Solution is $w = 30.17$

Certainty Equivalent (II)



Violations of Expected Utility Theory

Problems with EUT

A number of violations of expected utility have been discovered.

Alternative theories have been developed which seek to account for these violations.

$\rightarrow$ Best-known is Cumulative Prospect Theory of Daniel Kahneman and Amos Tversky.

Example 1

- Which of the two alternatives do you prefer?
  - A. 53% to win $50,000
  - B. 51% to win $60,000

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- Which of the two alternatives do you prefer?
  - A. sure gain of $50,000
  - B. 98% to win $60,000

Allais Paradox

Which of the two alternatives do you prefer?

- A. 53% to win \$50,000
- B. 51% to win \$60,000

Most people choose B

Which of the two alternatives do you prefer?

- A. sure gain of \$50,000
- B. 98% to win \$60,000

Most people choose A

$\Rightarrow$ Not consistent with expected utility

Allais Paradox: The Violation

51% to win \$60,000 is preferable to 53% to win \$50,000:

$$51\% \cdot v(60{,}000) \quad > \quad 53\% \cdot v(50{,}000)$$

If we add 47% to each probability:

$$\begin{aligned} 51\% \cdot v(60{,}000) \quad &> \quad 53\% \cdot v(50{,}000) \\ &+ \\ 47\% \cdot v(60{,}000) \quad &> \quad 47\% \cdot v(50{,}000) \end{aligned}$$

we have:

$$98\% \cdot v(60{,}000) > 100\% \cdot v(50{,}000)$$

$\rightarrow$ which is inconsistent with the observed choice (violation of Independence Axiom)

Example 2

Which of the two alternatives do you prefer?

- A. sure loss of $750
- B. 75% to lose $1,000 and 25% to lose nothing

Risk-Seeking for Losses

Which of the two alternatives do you prefer?

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- B. 75% to lose $1,000 and 25% to lose nothing

Most people choose B

→ People are risk-seeking in the domain of losses

Example 3

1. In addition to whatever you own, you have been given \$1,000. Now you are asked to choose one of the two options:
   - A. sure gain of \$250
   - B. 25% to get \$1,000 and 75% to get nothing

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2. In addition to whatever you own, you have been given \$2,000. Now you are asked to choose one of the two options:
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Reference Dependence

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   - B. 75% to lose \$1,000 and 25% to lose nothing

People choose A in 1 and B in 2, even though the outcomes are identical.

→ People evaluate losses and gains in different ways

Example 4

Would you accept the following gamble?

50% to win \$125, 50% to lose \$100

Losses Loom Larger Than Gains: Loss Aversion

Would you accept the following gamble?

"50% to win \$125, 50% to lose \$100"

Most people choose not to accept.

- Expected Utility Theory implies people should behave almost in a risk-neutral way for small gambles
- Rabin (2000) theorem: anyone who rejects a favorable gamble with small stakes is mathematically committed to a foolish level of risk aversion for larger gambles. E.g., if an individual does not accept the above gamble for any wealth, then they would also not accept "50% to win \$60,000, 50% to lose \$600"

→ People seem to be overly sensitive to losses

Loss Aversion: The Loss Aversion Ratio

Consider a 50:50 gamble in which you can lose $100:

- What is the smallest gain that I need to balance an equal chance to lose $100?

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- For most people: \$200

- Loss aversion ratio: It has been estimated in the range of 1.5–2.5
- It depends on the individual: when participants were asked to “think like a trader”, they became less loss-averse and emotional reaction to losses was lower

Loss Aversion: Dependence on Stakes

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3. What about a possible loss of $2,000?

- Loss aversion ratio tends to slightly increase when the stakes rise

Bernoulli's Error

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→ "The theory (EU) ignores the fact that utility depends on the history of one's wealth, not only on present wealth."

→ People evaluate with respect to a reference point

Conclusions

Conclusions

The vast majority of models assume that investors evaluate gambles according to the Expected Utility Framework (von Neumann and Morgenstern, 1944).

Preferences that satisfy a number of plausible axioms can be represented by the expectation of a utility function.

- Daniel Bernoulli (18th century): Expected Utility Theory—relationship between the psychological value or desirability of money and the actual amount of money
- Utility as a logarithmic function of wealth
- Marginal utility is decreasing with wealth $\rightarrow$ risk aversion
- People make choices to maximize expected utility

A number of violations of expected utility have been discovered.

Closing Quiz

Interactive quiz on Vevox

Test your understanding of:
Expected Utility Theory, attitudes to risk, and violations of EUT

Please open Vevox and enter the session code